IMPACT: International Journal of Research in
Humanities, Arts and Literature (IMPACT: IJRHAL)
ISSN (P): 2347-4564; ISSN (E): 2321-8878
Vol. 6, Issue 11, Nov 2018, 201-214

OSCILLATORY PROPERTIES OF A CLASS OF CONFORMABLE FRACTIONAL GENERALIZED LIENARD EQUATIONS

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Received: 13 Nov 2018 **Accepted:** 19 Nov 2018 **Published:** 27 Nov 2018

ABSTRACT

The main objective of the present article is to study the oscillatory behavior of conformable fractional generalized Lienard equations. We obtain some new sufficient conditions that guarantee all solutions are oscillatory by using Riccati transformation technique. Suitable examples are inserted in order to illustrate the effectiveness of our obtained results.

KEYWORDS: *Conformable Fractional Differential Equation, Generalized Lienard Equation, Oscillation*

INTRODUCTION

Fractional calculus is nowadays one of the most intensively developing areas of mathematical analysis, including several definitions of fractional operators like Riemann-Liouville, Caputo, and Grunwald-Letnikov. It has been shown in various studies that fractional-order models capture phenomena and properties that integer orders neglect. These non-integer derivatives have been widely applied in different branches such as application in a genetic algorithm, the planner in signal processing, a tensile and flexural strength of disorder materials in solid mechanics, biology and physics, we refer the books [1,8,11,13,14,17].

Recently, a new fractional derivative called the conformable fractional derivative is introduced which is based on the basic limit definition of the derivative in Khalil [9]. There are many papers have devoted to the conformable fractional derivative, see, for example [2,6,7], and the references cited therein.

In 1928, Lienard [10] investigated the sufficient conditions for the occurrence of auto-oscillations in the system governed by

$$x'' + f(x)x' + x = 0. \tag{1.1}$$

In 2012, Matinfar et al. [12] solved the Lienard equation of the form

$$u'' + f(u)u' + g(u) = h(x). \tag{1.2}$$

by differential transform method.

The Lienard equation is closely connected with the Rayleigh equation. For a particular case of (1.2), namely Van der Pol equation for the choices of $f(u) = \epsilon(u^2 - 1), g(u) = u$ and $h(x) = 0$. Van der Pol equation served as a nonlinear model of electronic oscillation. The Lienard equations are used to model the oscillating circuits emerging in radio and vacuum tube technology.

In 2013, Zeghdoudi et al. [16] considered the scalar Lienard equations

$$\ddot{x}(t) = f\big(x(t)\big)\dot{x}(t) + g\big(x(t)\big), x(t) \in \mathbb{R}. \tag{1.3}$$

Abdullah [3,4] studied the oscillation criteria for second-order nonlinear differential equations. In 2016, Abdullah [5] studied the oscillation of a class of Lienard equation of the form

$$\ddot{x}(t) + f\big(x(t)\big)\big(\dot{x}(t)\big)^2 + g\big(x(t)\big) = 0, t \geq t_0, \tag{1.4}$$

where f and g are continuously differentiable functions on R.

It seems that there has been no work done on the conformable fractional nonlinear Lienard differential equations. The work along this line is of great interest and which is the main motivation of our paper.

In this paper, we study the oscillatory behavior of the solutions of conformable fractional generalized Lienard equation of the form

$$T_\alpha\Big(r(t)T_\alpha\big(x(t)\big)\Big) + f\big(x(t)\big)\Big(T_\alpha\big(x(t)\big)\Big)^2 + g\big(x(t)\big) = 0, \;\; t \geq t_0, \tag{1.5}$$

where T_α denote the conformable fractional derivative with respect to α, $0 < \alpha \leq 1$.

We assume throughout this paper that :

(A_1) $r(t) \in C^\alpha\big([t_0,\infty),(0,\infty)\big), r(t)T_\alpha\big(x(t)\big) \in C^\alpha\big([t_0,\infty),(0,\infty)\big);$

(A_2) $f\big(x(t)\big)$ and $g\big(x(t)\big)$ are continuously differentiable functions on R.

Note that if r(t) = 1, then the equation (1.5) is reduced to the new class, called the conformable class of Lienard equation and in addition to that when $\alpha = 1$, the equation (1.5) reduces to the Lienard equation (1.4).

A nontrivial solution $x(t)$ of differential equation (1.5) is said to be oscillatory if it has arbitrarily large zeros otherwise it said to be nonoscillatory. The equation (1.5) is oscillatory if all its solutions are oscillatory.

This paper is organized as follows: In Section 2, we recall the basic definitions of the conformable fractional derivative. In Section 3, we present some new oscillation criteria for all solutions of generalized Lienard equation (1.5). In Section 4, examples are provided to illustrate our main results.

2. PRELIMINARIES

In this section, we shall present some preliminary results on conformable fractional derivative. First, we shall start with the definition.

Definition: 2.1 [9] Given a function $f: [0,\infty) \to \mathbb{R}$. Then the conformable fractional derivative of f of order α is defined by

$$T_\alpha (f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0,1]$. If f is α-differentiable in some $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

We will sometimes write $f^{(\alpha)}(t)$ for $T_\alpha (f)(t)$, to denote the conformable fractional derivatives of f of order α.

Some Properties of Conformable Fractional Derivative [9]:

Let $\alpha \in (0,1]$ and f and g be α-differentiable at a point $t > 0$.Then

(P_1) $T^\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.

(P_2) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

(P_3) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

(P_4) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

(P_5) If, in addition, f is differentiable, then $T_\alpha (f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.

3. MAIN RESULTS:

In this section, we establish several new sufficient conditions for the oscillation of solutions of (1.5) based on the Riccati transformation.

Theorem: 3.1. Assume that $(A_1) - (A_2)$ hold. If

$$\lim_{t\to\infty} \frac{1}{4} \int_{t_0}^{t} \left(4s^\alpha - \frac{(r(s))^2}{s^\alpha\left(f(x(s))g(x(s)) + r(s)\frac{dg(x(s))}{dx}\right)} \right) ds = \infty \tag{3.1}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{s^{\alpha-1}\left(f(x(s))g(x(s)) + r(s)\frac{dg(x(s))}{dx}\right)}{s(r(s))^2} ds = \infty. \tag{3.2}$$

Then every solution of (1.5) is oscillatory.

Proof:

Let $x(t)$ be a nonoscillatory solution of (1.5) on the interval $[t_1,\infty)$. Without loss of generality, its solution can be supposed such that $x(t) > 0$ on $[t_1,\infty)$. Define the generalized Riccati substitution

$$w(t) = -\frac{t\left(r(t)T_\alpha\left(x(t)\right)\right)}{g\left(x(t)\right)}, t \geq t_1. \tag{3.3}$$

Then $w(t)$ is well defined.

$$T_\alpha\left(w(t)\right) = \frac{-g\left(x(t)\right)\left[t^{1-\alpha}\left(r(t)T_\alpha\left(x(t)\right)\right)+tT_\alpha\left(r(t)T_\alpha\left(x(t)\right)\right)\right]+t\left(r(t)T_\alpha\left(x(t)\right)\right)t^{1-\alpha}\frac{dg(x(t))}{dx}x'(t)}{\left(g\left(x(t)\right)\right)^2}$$

$$t^{1-\alpha}w'(t) = t^{1-\alpha}\frac{w(t)}{t} + f\left(x(t)\right)g\left(x(t)\right)\frac{(w(t))^2}{t(r(t))^2} + t + \frac{dg\left(x(t)\right)}{dx}\frac{(w(t))^2}{tr(t)}$$

$$w'(t) = \frac{t^{\alpha-1}\left(f\left(x(t)\right)g\left(x(t)\right)+r(t)\frac{dg(x(t))}{dx}\right)}{t(r(t))^2}\left[(w(t))^2 + \frac{w(t)(r(t))^2}{t^{\alpha-1}\left(f\left(x(t)\right)g\left(x(t)\right)+r(t)\frac{dg(x(t))}{dx}\right)}\right] + t^\alpha$$

$$w'(t) = \frac{t^{\alpha-1}\left(f\left(x(t)\right)g\left(x(t)\right)+r(t)\frac{dg(x(t))}{dx}\right)}{t\left(r(t)\right)^2}\left[\left(w(t) + \frac{\left(r(t)\right)^2}{2t^{\alpha-1}\left(f\left(x(t)\right)g\left(x(t)\right)+r(t)\frac{dg(x(t))}{dx}\right)}\right)^2\right]$$

$$+t^\alpha - \frac{(r(t))^2}{4t^\alpha\left(f\left(x(t)\right)g\left(x(t)\right)+r(t)\frac{dg(x(t))}{dx}\right)}. \tag{3.4}$$

Integrating both sides of the above equation from t_0 to t, we have

$$w(t) = w(t_0) + \int_{t_0}^{t}\frac{s^{\alpha-1}\left(f\left(x(s)\right)g\left(x(s)\right)+r(s)\frac{dg(x(s))}{dx}\right)}{sr(s)^2}\left[\left(w(s) + \frac{\left(r(s)\right)^2}{2s^{\alpha-1}\left(f\left(x(s)\right)g\left(x(s)\right)+r(s)\frac{dg(x(s))}{dx}\right)}\right)^2\right]ds$$

$$+\int_{t_0}^{t}\left(s^\alpha - \frac{(r(s))^2}{4s^\alpha\left(f\left(x(s)\right)g\left(x(s)\right)+r(s)\frac{dg(x(s))}{dx}\right)}\right)ds.$$

By using the hypothesis (3.1) implies there exist $t_1 \geq t_0$ such that

$$w(t) \geq \int_{t_1}^{t}\frac{s^{\alpha-1}\left(f\left(x(s)\right)g\left(x(s)\right)+r(s)\frac{dg\left(x(s)\right)}{dx}\right)}{sr(s)^2}$$

$$\times\left[\left(w(s)+\frac{(r(s))^2}{2s^{\alpha-1}\left(f(x(s))g(x(s))+r(s)\frac{dg(x(s))}{dx}\right)}\right)^2\right]ds.$$

Consider

$$Q(t)=\int_{t_1}^{t}\frac{s^{\alpha-1}\left(f(x(s))g(x(s))+r(s)\frac{dg(x(s))}{dx}\right)}{s\,r(s)^2}\left[\left(w(s)+\frac{(r(s))^2}{2s^{\alpha-1}\left(f(x(s))g(x(s))+r(s)\frac{dg(x(s))}{dx}\right)}\right)^2\right]ds, \quad (3.5)$$

then we have $w(t)\geq Q(t)>0$. Differentiating the above, we get

$$Q'(t)=\frac{t^{\alpha-1}\left(f(x(t))g(x(t))+r(t)\frac{dg(x(t))}{dx}\right)}{t(r(t))^2}$$

$$\times\left[\left(w(t)+\frac{(r(t))^2}{2t^{\alpha-1}\left(f(x(t))g(x(t))+r(t)\frac{dg(x(t))}{dx}\right)}\right)^2\right]$$

$$\geq\frac{t^{\alpha-1}\left(f(x(t))g(x(t))+r(t)\frac{dg(x(t))}{dx}\right)}{t(r(t))^2}\left[\left(Q(t)+\frac{(r(t))^2}{2t^{\alpha-1}\left(f(x(t))g(x(t))+r(t)\frac{dg(x(t))}{dx}\right)}\right)^2\right]$$

$$\geq\frac{t^{\alpha-1}\left(f(x(t))g(x(t))+r(t)\frac{dg(x(t))}{dx}\right)}{t(r(t))^2}(Q(t))^2.$$

Therefore,

$$\frac{t^{\alpha-1}\left(f(x(t))g(x(t))+r(t)\frac{dg(x(t))}{dx}\right)}{t(r(t))^2}\leq\frac{Q'(t)}{(Q(t))^2}.$$

Integrating both sides of this inequality from t_1 to t for $t\geq t_1$, we get

$$\int_{t_1}^{t}\frac{s^{\alpha-1}\left(f(x(s))g(x(s))+r(s)\frac{dg(x(s))}{dx}\right)}{t(r(s))^2}ds\leq\frac{1}{Q(t_1)}-\frac{1}{Q(t)},$$

since $Q(t)>0$. Thus

$$\lim_{t\to\infty}\int_{t_1}^{t}\frac{s^{\alpha-1}\left(f\big(x(s)\big)g\big(x(s)\big)+r(s)\frac{dg\left(x(s)\right)}{dx}\right)}{t\big(r(s)\big)^2}ds<\frac{1}{Q(t_1)},$$

which contradicts (3.2). Hence the differential equation (1.5) is oscillatory.

Theorem: 3.2. Assume that $(A_1)-(A_2)$ hold. If

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{s^{\alpha-1}\left(f\big(x(s)\big)g\big(x(s)\big)+r(s)\frac{dg(x(s))}{dx}\right)}{\big(r(s)\big)^2}ds=\infty. \qquad (3.6)$$

Then every solution of (1.5) is oscillatory.

Proof:

Assume that $x(t)$ is a nonoscillatory solution of (1.5). Without loss of generality we may assume that $x(t)$ is an eventually positive solution of (1.5). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$.

Consider the Riccati transformation

$$u(t)=-\frac{r(t)T_\alpha\big(x(t)\big)}{g\big(x(t)\big)},\ t\geq t_1. \qquad (3.7)$$

Then u(t) is well defined and differentiating α-times with respect to 't', we have

$$T_\alpha u(t)=\frac{-g\big(x(t)\big)T_\alpha\Big(r(t)T_\alpha\big(x(t)\big)\Big)+r(t)T_\alpha\big(x(t)\big)T_\alpha\big(g(x(t))\big)}{\big(g(x(t))\big)^2}$$

$$u'(t)=t^{\alpha-1}\left(\frac{\left(f\big(x(t)\big)g\big(x(t)\big)+r(t)\frac{dg\left(x(t)\right)}{dx}\right)}{\big(r(t)\big)^2}\right)\big(u(t)\big)^2+t^{\alpha-1}$$

Integrating the above from t_0 to t, we get

$$u(t)=u(t_0)+\int_{t_0}^{t}\frac{s^{\alpha-1}\left(f\big(x(s)\big)g\big(x(s)\big)+r(s)\frac{dg\left(x(s)\right)}{dx}\right)}{\big(r(s)\big)^2}(u(s))^2\,ds+\int_{t_0}^{t}s^{\alpha-1}ds$$

$$u(t)\geq u(t_0)+\int_{t_0}^{t}\frac{s^{\alpha-1}\left(f\big(x(s)\big)g\big(x(s)\big)+r(s)\frac{dg\left(x(s)\right)}{dx}\right)}{\big(r(s)\big)^2}(u(s))^2\,ds+t^{\alpha-1}\int_{t_0}^{t}ds.$$

Then for some $t_1 \geq t_0$, we have

$$u(t) \geq \int_{t_1}^{t} \frac{s^{\alpha-1}\left(f(x(s))g(x(s))+r(s)\frac{dg(x(s))}{dx}\right)}{(r(s))^2}(u(s))^2\,ds.$$

Let $R(t)$ for $t \geq t_1$ by

$$R(t) = \int_{t_1}^{t} \frac{s^{\alpha-1}\left(f(x(s))g(x(s))+r(s)\frac{dg(x(s))}{dx}\right)}{(r(s))^2}(u(s))^2\,ds, \tag{3.8}$$

then we have $u(t) \geq R(t) > 0$. Differentiating (3.8), we get

$$R'(t) = \frac{t^{\alpha-1}\left(f(x(t))g(x(t))+r(t)\frac{dg(x(t))}{dx}\right)}{(r(t))^2}(u(t))^2$$

$$\geq \frac{t^{\alpha-1}\left(f(x(t))g(x(t))+r(t)\frac{dg(x(t))}{dx}\right)}{(r(t))^2}(R(t))^2.$$

Thus

$$\frac{t^{\alpha-1}\left(f(x(t))g(x(t))+r(t)\frac{dg(x(t))}{dx}\right)}{(r(t))^2} \leq \frac{R'(t)}{(R(t))^2}.$$

Integrating the above inequality from t_1 to t, $R(t) > 0$,we get

$$\int_{t_1}^{t} \frac{s^{\alpha-1}\left(f(x(s))g(x(s))+r(s)\frac{dg(x(s))}{dx}\right)}{(r(s))^2}ds \leq \frac{1}{R(t_1)} - \frac{1}{R(t)},$$

We conclude that

$$\lim_{t\to\infty}\int_{t_1}^{t} \frac{s^{\alpha-1}\left(f(x(s))g(x(s))+r(s)\frac{dg(x(s))}{dx}\right)}{(r(s))^2}ds < \frac{1}{R(t_1)},$$

we obtain a contradiction to (3.6). This completes the proof.

Theorem: 3.3. Assume that $(A_1) - (A_2)$ hold. If

$$\lim_{t\to\infty}\int_{t_0}^{t} s^{\alpha-1}\frac{g(x(s))}{f(x(s))}ds = \infty \tag{3.9}$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{s^{\alpha-1}\left(\left(f(x(s))\right)^2+r(s)\frac{df(x(s))}{dx}\right)}{\left(r(s)\right)^2}ds=\infty. \quad (3.10)$$

Then every solution of (1.5) is oscillatory.

Proof:

Suppose that x(t) is a nonoscillatory solution of Equation (1.5). We may assume without loss of generality that $x(t)\neq 0$ with $t\geq t_1$. Define the function

$$\phi(t)=-\frac{r(t)T_\alpha(x(t))}{f(x(t))},\ t\geq t_1.$$

Then $\phi(t)$ is well defined.

$$T_\alpha\phi(t)=\frac{-f(x(t))T_\alpha\left(r(t)T_\alpha(x(t))\right)+r(t)T_\alpha(x(t))T_\alpha(f(x(t)))}{(f(x(t)))^2}$$

$$\phi'(t)=\left(\frac{t^{\alpha-1}\left(\left(f(x(t))\right)^2+r(t)\frac{df(x(t))}{dx}\right)}{\left(r(t)\right)^2}\right)\left(\phi(t)\right)^2+t^{\alpha-1}\frac{g(x(t))}{f(x(t))}. \quad (3.11)$$

Integrating the above from t_0 to t, we have

$$\phi(t)=\phi(t_0)+\int_{t_0}^{t}\frac{s^{\alpha-1}\left(\left(f(x(s))\right)^2+r(s)\frac{df(x(s))}{dx}\right)}{\left(r(s)\right)^2}(\phi(s))^2ds+\int_{t_0}^{t}s^{\alpha-1}\frac{g(x(s))}{f(x(s))}ds.$$

Now, using (3.9), we can choose t_1 sufficiently large so that

$$\phi(t)\geq\int_{t_1}^{t}\frac{s^{\alpha-1}\left(\left(f(x(s))\right)^2+r(s)\frac{df(x(s))}{dx}\right)}{\left(r(s)\right)^2}(\phi(s))^2ds.$$

Let us consider $H(t)$ for $t\geq t_1$ by

$$H(t)=\int_{t_1}^{t}\frac{s^{\alpha-1}\left(\left(f(x(s))\right)^2+r(s)\frac{df(x(s))}{dx}\right)}{\left(r(s)\right)^2}(\phi(s))^2ds. \quad (3.12)$$

Then we have $\phi(t)\geq H(t)>0$. Differentiating (3.12), we obtain

$$H'(t)=\frac{t^{\alpha-1}\left(\left(f(x(t))\right)^2+r(t)\frac{df(x(t))}{dx}\right)}{\left(r(s)\right)^2}(\phi(t))^2$$

$$\geq\frac{t^{\alpha-1}\left(\left(f(x(t))\right)^2+r(s)\frac{df(x(t))}{dx}\right)}{\left(r(s)\right)^2}(H(t))^2.$$

Therefore,

$$\frac{t^{\alpha-1}\left(\left(f(x(t))\right)^2 + r(t)\frac{df(x(t))}{dx}\right)}{\left(r(t)\right)^2} \le \frac{H'(t)}{(H(t))^2}.$$

Integrating from t_1 to t for $t \ge t_1$ with $H(t) > 0$, we

$$\int_{t_1}^{t} \frac{s^{\alpha-1}\left(\left(f(x(s))\right)^2 + r(s)\frac{df(x(s))}{dx}\right)}{\left(r(s)\right)^2} ds \le \frac{1}{H(t_1)} - \frac{1}{H(t)},$$

we conclude that

$$\lim_{t\to\infty} \int_{t_1}^{t} \frac{s^{\alpha-1}\left(\left(f(x(s))\right)^2 + r(s)\frac{df(x(s))}{dx}\right)}{\left(r(s)\right)^2} ds < \frac{1}{H(t_1)}.$$

This contradicts the assumption (3.10).Hence, the proof is completed.

Theorem:3.4. Assume that $(A_1) - (A_2)$ hold. If for some function $\delta(t) \in C^{\alpha}([t_0,\infty),(0,\infty))$, for all sufficiently large $t_1 \ge t_0$ such that

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{s^{\alpha-1}\left(kr(s) + \left(f(x(s))\right)^2\right)}{\delta(s)(r(s))^2} ds = \infty \tag{3.13}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} s^{\alpha-1} \frac{\delta(s) g(x(s))}{f(x(s))} - \frac{\left(\delta'(s)\right)^2 (r(s))^2}{4\delta(s) s^{\alpha-1}\left(kr(s) + \left(f(x(s))\right)^2\right)} ds = \infty. \tag{3.14}$$

Then every solution of (1.5) is oscillatory.

Proof:

Let x(t) be a nonoscillatory solution of (1.5).Then there exists a $t_1 \ge t_0$ such that $x(t) \ne 0$ for all $t \ge t_1$. Without loss of generality, we may assume that $x(t) > 0$ on the interval $[t_1, \infty)$.

Defining a generalized Riccati transformation by

$$\psi(t) = -\delta(t)\frac{r(t)T_\alpha(x(t))}{f(x(t))}, \; t \ge t_1. \tag{3.15}$$

Then $\psi(t)$ is well defined and differentiating,

$$T_\alpha \psi(t) = \frac{-f(x(t))\left[t^{1-\alpha}\delta'(t) r(t) T_\alpha(x(t)) + \delta(t) T_\alpha\left(r(t) T_\alpha(x(t))\right)\right] + \delta(t) r(t) T_\alpha(x(t)) t^{1-\alpha} f'(x(t)) x'(t)}{\left(f(x(t))\right)^2}$$

$$\psi'(t) = \left(\frac{t^{\alpha-1}\left(r(t)f'(x(t)) + (f(x(t)))^2\right)}{\delta(t)(r(t))^2}\right)(\psi(t))^2 + \delta'(t)\frac{\psi(t)}{\delta(t)} + +t^{\alpha-1}\frac{\delta(t)g(x(t))}{f(x(t))} \tag{3.16}$$

and using $f'(x) \geq k > 0$ where k is a constant,

$$\geq \frac{t^{\alpha-1}\left(kr(t) + (f(x(t)))^2\right)}{\delta(t)(r(t))^2}\left[\left(\psi(t) + \frac{\delta'(t)(r(t))^2}{2t^{\alpha-1}\left(kr(t) + (f(x(t)))^2\right)}\right)^2\right] +$$

$$t^{\alpha-1}\frac{\delta(t)g(x(t))}{f(x(t))} - \frac{(r(t))^2(\delta'(t))^2}{4t^{\alpha-1}\delta(t)\left(kr(t) + (f(x(t)))^2\right)}$$

Integrating both sides of the above equation from t_0 to t, we get

$$\psi(t)$$

$$\geq \psi(t_0) + \int_{t_0}^{t} \frac{s^{\alpha-1}\left(kr(s) + (f(x(s)))^2\right)}{\delta(s)(r(s))^2}\left[\left(\psi(s) + \frac{\delta'(s)(r(s))^2}{2s^{\alpha-1}\left(kr(s) + (f(x(s)))^2\right)}\right)^2\right] ds$$

$$+ \int_{t_0}^{t}\left(s^{\alpha-1}\frac{\delta(s)g(x(s))}{f(x(s))} - \frac{(r(s))^2(\delta'(s))^2}{4s^{\alpha-1}\delta(s)\left(kr(s) + (f(x(s)))^2\right)}\right) ds$$

By (3.14), we have that

$$\psi(t) \geq \int_{t_1}^{t} \frac{s^{\alpha-1}\left(kr(s) + (f(x(s)))^2\right)}{\delta(s)(r(s))^2}\left[\left(\psi(s) + \frac{\delta'(s)(r(s))^2}{2s^{\alpha-1}\left(kr(s) + (f(x(s)))^2\right)}\right)^2\right] ds.$$

Define a function $M(t)$ for $t \geq t_1$ by

$$M(t) = \int_{t_1}^{t} \frac{s^{\alpha-1}\left(kr(s) + (f(x(s)))^2\right)}{\delta(s)(r(s))^2}\left[\left(\psi(s) + \frac{\delta'(s)(r(s))^2}{2s^{\alpha-1}\left(kr(s) + (f(x(s)))^2\right)}\right)^2\right] ds, \tag{3.17}$$

then we have $\psi(t) \geq M(t) > 0$.

$$M'(t) = \frac{t^{\alpha-1}\left(kr(t) + (f(x(t)))^2\right)}{\delta(t)(r(t))^2}\left[\left(\psi(t) + \frac{\delta'(t)(r(t))^2}{2t^{\alpha-1}\left(kr(t) + (f(x(t)))^2\right)}\right)^2\right]$$

$$\geq \frac{t^{\alpha-1}\left(kr(t)+\left(f(x(t))\right)^2\right)}{\delta(t)\left(r(t)\right)^2}\left[\left(M(t)+\frac{\delta'(t)\left(r(t)\right)^2}{2t^{\alpha-1}\left(kr(t)+\left(f(x(t))\right)^2\right)}\right)^2\right]$$

$$\geq \frac{t^{\alpha-1}\left(kr(t)+\left(f(x(t))\right)^2\right)}{\delta(t)\left(r(t)\right)^2}\left(M(t)\right)^2.$$

Hence

$$\frac{t^{\alpha-1}\left(kr(t)+\left(f(x(t))\right)^2\right)}{\delta(t)\left(r(t)\right)^2} \leq \frac{M'(t)}{(M(t))^2}. \tag{3.18}$$

Integrating (3.18) from t_1 to t for $t \geq t_1$, we get

$$\int_{t_1}^{t} \frac{s^{\alpha-1}\left(kr(s)+\left(f(x(s))\right)^2\right)}{\delta(s)\left(r(s)\right)^2} ds \leq \frac{1}{M(t_1)} - \frac{1}{M(t)},$$

since $M(t) > 0$. Therefore

$$\lim_{t\to\infty} \int_{t_1}^{t} \frac{s^{\alpha-1}\left(kr(s)+\left(f(x(s))\right)^2\right)}{\delta(s)\left(r(s)\right)^2} ds \leq \frac{1}{M(t_1)},$$

which contradicts assumption (3.13), so (1.2) is oscillatory. Hence the proof of the theorem is complete.

4. EXAMPLES

Example 4.1 Consider the conformable fractional differential equation

$$T_{\frac{1}{2}}\left(\frac{1}{t}T_{\frac{1}{2}}(x(t))\right) + \frac{1-2x}{t(1+x^2)}\left(T_{\frac{1}{2}}(x(t))\right)^2 + (1+x^2) = 0, t \geq t_0. \tag{4.1}$$

Here $\alpha = \frac{1}{2}$, $r(t) = \frac{1}{t}$, $f(x(t)) = \frac{1-2x}{t(1+x^2)}$ and $g(x(t)) = 1 + x^2$. Now,

$$\lim_{t\to\infty} \frac{1}{4}\int_{t_0}^{t} \left(4s^{\alpha} - \frac{(r(s))^2}{s^{\alpha}\left(f(x(s))g(x(s))+r(s)\frac{dg(x(s))}{dx}\right)}\right) ds$$

$$= \lim_{t\to\infty} \int_{t_0}^{t} \left(s^{\frac{1}{2}} ds - \frac{1}{4}\int_{t_0}^{t} s^{-\frac{3}{2}} ds\right) \longrightarrow \infty \text{ as } t \longrightarrow \infty$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{s^{\alpha-1}\left(f(x(s))g(x(s)) + r(s)\frac{dg(x(s))}{dx}\right)}{s(r(s))^2} ds$$

$=\lim_{t\to\infty}\int_{t_0}^{t} s^{-\frac{1}{2}}\,ds \longrightarrow \infty$ as $t \longrightarrow \infty$.

Hence all the conditions of Theorem 3.1 are satisfied. Therefore, the differential equation (4.1) is oscillatory.

Example 4.2 Consider the fractional differential equation of the type

$$T_{\frac{1}{3}}\left(T_{\frac{1}{3}}(x(t))\right) + \cot(x(t))\left(T_{\frac{1}{3}}(x(t))\right)^2 - \cot(x(t)) = 0, t \geq t_0. \quad (4.2)$$

Here $\alpha = \frac{1}{3}$, $r(t) = 1$, $f(x(t)) = \cot(x(t))$ and $g(x(t)) = -\cot(x(t))$. Now,

$$\lim_{t\to\infty}\int_{t_0}^{t} \frac{s^{\alpha-1}\left(f(x(s))g(x(s)) + r(s)\frac{dg(x(s))}{dx}\right)}{(r(s))^2}\,ds$$

$$= \lim_{t\to\infty}\int_{t_0}^{t} s^{-\frac{2}{3}}\left(-(\cot(x(s)))^2 + (\operatorname{cosec}(x(s)))^2\right)ds$$

$$= \lim_{t\to\infty}\int_{t_0}^{t} s^{-\frac{2}{3}}\,ds \longrightarrow \infty \text{ as } t \longrightarrow \infty.$$

Therefore, Theorem 3.2 implies that the differential equation (4.2) is oscillatory.

Example 4.3 Consider the following conformable fractional differential equation

$$T_{\frac{1}{4}}\left(\frac{1}{t}T_{\frac{1}{4}}(x(t))\right) + t\left(T_{\frac{1}{4}}(x(t))\right)^2 + t^2 = 0, t \geq t_o, \quad (4.3)$$

Here $\alpha = \frac{1}{4}$, $r(t) = \frac{1}{t}$, $f(x(t)) = t$ and $g(x(t)) = t^2$.

Now,

$$\lim_{t\to\infty}\int_{t_0}^{t} s^{\alpha-1}\frac{g(x(s))}{f(x(s))}\,ds = \lim_{t\to\infty}\int_{t_0}^{t} s^{-\frac{3}{4}}\frac{s^2}{s}\,ds \longrightarrow \infty \text{ as } t \longrightarrow \infty$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t} \frac{s^{\alpha-1}\left((f(x(s)))^2 + r(s)\frac{df(x(s))}{dx}\right)}{(r(s))^2}\,ds = \lim_{t\to\infty}\int_{t_0}^{t} \frac{s^{-\frac{3}{4}}\left(s^2 + \frac{1}{s}\right)}{\frac{1}{s^2}}\,ds \longrightarrow \infty \text{ as } t \longrightarrow \infty.$$

Hence the differential equation (4.3) is oscillatory, conditions of Theorem 3.3 are verified.

Example 4.4 Consider the conformable fractional differential equation

$$T_{\frac{2}{3}}\left(\frac{1}{t}T_{\frac{2}{3}}(x(t))\right) + 2t\left(T_{\frac{2}{3}}(x(t))\right)^2 + 2t = 0, t \geq t_o, \quad (4.4)$$

Here $\alpha = \frac{2}{3}$, $r(t) = \frac{1}{t}$, $\delta(t) = t^2$ and $f\big(x(t)\big) = g\big(x(t)\big) = 2t$. Now,

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{s^{\alpha-1}\left(kr(s)+\left(f(x(s))\right)^2\right)}{\delta(s)(r(s))^2}\, ds = \lim_{t\to\infty} \int_{t_0}^{t} ks^{-\frac{4}{3}} + 4s^{\frac{5}{3}}\, ds \longrightarrow \infty \text{ as } t \longrightarrow \infty$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} s^{\alpha-1} \frac{\delta(s)g(x(s))}{f(x(s))} - \frac{\left(\delta'(s)\right)^2 (r(s))^2}{4s^{\alpha-1}\delta(s)\left(kr(s)+\left(f(x(s))\right)^2\right)}\, ds = \lim_{t\to\infty} \int_{t_0}^{t} s^{\frac{5}{3}} - \frac{1}{ks^{\frac{2}{3}}+4s^{\frac{11}{3}}}\, ds \longrightarrow \infty \text{ as } t \longrightarrow \infty.$$

By Theorem 3.4, Equation (4.4) is oscillatory.

Remark: All the results obtained in this paper can be extended to a forced generalized Lienard equation of the form

$$T_\alpha\Big(a(t)T_\alpha\big(y(t)\big)\Big) + g\big(y(t)\big)\varphi\Big(T_\alpha\big(y(t)\big)\Big) + f\big(y(t)\big) = e(t),\ t \geq t_0.$$

CONCLUSIONS

In this study, we have obtained some new oscillation results for some class of conformable fractional nonlinear Lienard differential equations by using Riccati technique. This work extends some of the results in the exiting classical literature [3,4,5] to the conformable fractional case.

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