IMPACT: International Journal of Research in Applied, Natural and Social Sciences (IMPACT: IJRANSS)
ISSN(E): 2321-8851; ISSN(P): 2347-4580
Vol. 2, Issue 7, Jul 2014, 35-46

DISCRETE QUARTIC SPLINE INTERPOLATION

Y. P. DUBEY & K. K. NIGAM

Department of Mathematics, L. N. C. T, Jabalpur, Madhya Pradesh, India

ABSTRACT

In this paper, we have obtained existence, uniqueness and error bounds for deficient discrete quartic spline interpolation.

KEYWORDS: Deficient, Discrete, Quartic Spline, Interpolation, Error Bounds

Subject Classification Code: 41AO5, 65DO7

1. INTRODUCTION

Let us consider a mesh on [0, 1] which is define by

$$0 = x_0 < x_1 < ...... < x_n = 1 \text{ with } x_i - x_{i-1} = P_i \qquad \text{for i = 1,2,....n.}$$

and h > 0, will be a real number, consider a real continuous function s(x,h) defined over [0, 1] which is such that its restriction s_i $on \lfloor x_{i-1}, x_i \rfloor$ is a polynomial of degree 4 or less for i = 1, 2,....n. Then s(x, h) defines a discrete quartic spline if

$$D_h^{\{1\}} s_i(x_i, h) = D_h^{\{j\}} s_{i+1}(x_i, h) \; j = 0,1 \tag{1.1}$$

Where the difference operator D_h are defined as

$$D_h^{\{0\}} f\ (x) = f(x)$$

$$D_h^{\{1\}} f(x) = \frac{f(x+h) - f(x)}{h}$$

$$D_h^{\{2\}} f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{2h}$$

Let $D(4,1,\Delta,h)$ is the class of deficient discrete quartic spline interpolation of deficiency one, where in $D*(4,1,\Delta,h)$ denotes the class of all discrete deficient quartic splines which satisfies the boundary condition

$$s(x_0, h) = f(x_0, h)$$

$$s(x_n, h) = f(x_n, h) \tag{1.2}$$

Mangasarian and Schumaker [6, 7] introduced discrete quartic splines to find minimization problem. Existence, uniqueness and convergence properties of discrete cubic spline interpolation matching the given function at mesh point have been studied by Lyche [4, 5] which have been generalized by Dikshit and Power [1] (see also Dikshit and Rana [2]). It has been by Baneva, Kendall and Stefanov [3] that the local behaviour of the derivative of a cubic spline interpolator is some times used to smooth a histogram which has been estimated in [8], Rana [8] has obtained a precise estimate concerning the discrete cubic spline interpolating the given function at the mesh points (See also [9], [10]). In the present paper we obtain a similar precise estimate concerning the deficient quartic spline interpolant matching the given function at two intermediate points between successive mesh points and first difference of interior points of interval [0, 1].

2. EXISTENCE AND UNIQUENESS

We introduced the following interpolating conditions for a given function *f*.

$$s(\alpha_i)=f(\alpha_i) \tag{2.1}$$

$$s(\beta_i)=f(\beta_i) \tag{2.2}$$

$$D_h^{\{1\}}s(\gamma_i)=D_h^{\{1\}}f(\gamma_i) \tag{2.3}$$

Where $\alpha_i=x_{i-1}+\frac{1}{3}P_i=\gamma_i$, for i = 1, 2,....n.

$$\beta_1=x_{i-1}+\frac{1}{2}P_i$$

we shall prove the following.

Theorem 2.1

Let f be a 1-periodic, then for any h>0 then these exist a unique 1-periodic deficient discrete quartic spline s in the class $D*(4,1,\Delta,h)$ which satisfies interpolatory condition (2.1) - (2.3).

Proof

Let P(z) be a quartic polynomial on [0, 1], then we can show that

$$P(z)=P\left(\frac{1}{3}\right).q_1(z)+P\left(\frac{1}{2}\right)q_2(z)+D_h^{\{1\}}P\left(\frac{1}{3}\right)q_3(z)+P(0)q_4(z)+P(1)q_5(z) \tag{2.4}$$

Where $$q_1(z)=\frac{\left[\left(\frac{1}{6}+\frac{3}{2}h^2\right)z+\left(2-\frac{9}{2}h^2\right)z^2+z^3\left(-\frac{29}{6}+24h^2\right)+\left(3-18h^2\right)z^4\right]}{\left(\frac{2}{81}-\frac{1}{3}h^2\right)}$$

$$q_2(z)=\frac{\left[\frac{32}{81}z+\left(\frac{-224}{81}+\frac{16}{3}h^2\right)z^2+z^3\left(\frac{160}{27}-\frac{64}{3}h^2\right)+z^4\left(-\frac{32}{9}+16h^2\right)\right]}{\left(\frac{2}{81}-\frac{1}{3}h^2\right)}$$

$$q_3(z)=\frac{\left[\frac{-1}{9}z+\frac{2}{3}z^2-\frac{11}{9}z^3+\frac{2}{3}z^4\right]}{\left(\frac{2}{81}-\frac{1}{3}h^2\right)}$$

$$q_4(z)=\frac{\left[1+\left(-\frac{2}{27}+\frac{4}{3}h^2\right)z+\left(\frac{58}{81}+\frac{h^2}{3}\right)z^2-\left(\frac{26}{27}+\frac{16}{3}h^2\right)z^3+\left(\frac{4}{9}+4h^2\right)z^4\right]}{\left(\frac{2}{81}-\frac{h^2}{3}\right)}$$

$$q_5(z)=\frac{\left[\left(-\frac{1}{162}+\frac{1}{6}h^2\right)z+\left(\frac{8}{162}-\frac{7}{6}h^2\right)z^2+\left(-\frac{7}{54}+\frac{8}{3}h^2\right)z^3+\left(\frac{1}{9}-2h^2\right)z^4\right]}{\left(\frac{2}{81}-\frac{h^2}{3}\right)}$$

Denoting $t=\frac{x-x_i}{p_i}, 0\le t\le 1$, we can expressed (2.4) in the form of restriction $s_i(x,h)$ of the deficient discrete quartic spline s(x, h) on $[x_{i-1},x_{i+1}]$ as follows :-

$$s(x,h)=f(\alpha_i)\,q_1(x)+f(\beta_i)\,q_2(x)+P_i\,D_h^{\{1\}}(\gamma_i)q_3(x)$$

$$+s_i(x)\,q_4(x)+s_{i+1}(x)\,q_5(x) \tag{2.5}$$

Observing (2.5) it may easily be verify that $s_i(x,h)$ is a quartic on $[x_i,x_{i+1}]$ for i=0, 1,...., n-1 satisfying (2.1) - (2.3) and writing $H(a,b)=a+bh^2$, for real a, b we shall apply continuity of the first difference of $s(x,h)$ at x_i in (2.5) to see that

$$P_1^3\left[H\left(\frac{92}{81},2\right)P_{i-1}^2+H\left(\frac{2326}{189},\frac{-32}{3}\right)h^2\right]s_{i-1}$$

$$+\left[P_i^3\left\{H\left(\frac{-4}{27},\frac{13}{6}\right)P_{i-1}^2+H\left(\frac{-17}{54},\frac{16}{3}\right)h^2\right\}\right.$$

$$\left.+P_{i-1}^3\left\{H\left(\frac{-2}{9},\frac{4}{3}\right)P_i^2-H\left(\frac{26}{27},\frac{16}{3}\right)h^2\right\}\right]s_i$$

$$+P_{i-1}^3\left\{H\left(-\frac{1}{162},\frac{1}{6}\right)P_i^2+H\left(\frac{-7}{54},\frac{8}{3}\right)h^2\right\}s_{i+1} \tag{2.6}$$

$=F_i$ i = 1, 2,.....n-1.

Where $F_1=P_i^3\left[\left\{H\left(\frac{-37}{6},59\right)P_{i-1}^2+h^2\,H\left(\frac{-569}{6}-48h^2\right)\right]\right.$

$$f(\alpha_{i-1})-P_{i-1}^3\left\{H\left(\frac{-1}{6},\frac{-3}{2}\right)P_i^2+H\left(\frac{-29}{6},24\right)h^2\right\}$$

$$f(\alpha_i)+P_i^3\left\{-H\left(\frac{128}{81},\frac{736}{9}\right)P_{i-1}^2+H\left(\frac{-224}{27},\frac{128}{3}\right)h^2\right\}$$

$$f(\beta_{i-1})-f(\beta_i)\left\{-\left(H\frac{32}{81},0\right)P_i^2+H\left(\frac{160}{27},\frac{-64}{3}\right)h^2\right\}$$

$$+P_i^3\,P_{i-1}\,H\left(\frac{2}{9},\frac{13}{9}\right)D_h^{\{1\}}\,f(\gamma_{i-1})+D_h^{\{1\}}\,f(\gamma_i)P_{i-1}^3P_i\,H\left(\frac{1}{9},\frac{11}{9}\right)]$$

Existence, uniqueness of s(x, h) depend on the existence of a unique solution of set of equation (2.6). It is easy to observe that in (2.6) absolute value of the coefficient of s_i dominates over the sum of the absolute values of the coefficient of s_{i+1} and s_{i-1} i.e. is positive.

$$T_i(P,h)=\left[H\left(\frac{80}{81},\frac{1}{6}\right)P_{i-1}^2+H\left(\frac{1511}{126},\frac{16}{3}\right)h^2\ +H\left(\frac{35}{162},\frac{7}{6}\right)P_i^2+H\left(\frac{45}{54},\frac{8}{3}\right)h^2\right]$$

Thus the coefficient matrix of equation (2.6) is diagonally dominant and hence invertible.

Remark: In the case $h\to 0$ theorem 2.1 gives the corresponding result for continuous quartic spline interpolation under condition (2.1) - (2.3).

3. ERROR BOUNDS

Now system of equation (2.8) may be written as

$$A(h)\ M(h)=F$$

Where A(h) is coefficient matrix and $M(h)=s_i(h)$. However, as already shown in the proof of theorem 2.1. *A(h)* is invertible. Denoting the inverse of *A(h)* by $A^{-1}(h)$ we note that row max norm $A^{-1}(h)$ satisfies the following inequality :-

$$\|A^{-1}(h)\|\le y(h) \tag{3.1}$$

Where $y(h)=\max\{T_i(h)\}^{-1}$. For convenience we assure in this section that 1 = Nh when N is positive integer. It is also assume that the mesh points $\{x_i\}$ are such that $x_i \in [0,1]_h$ for i = 0, 1.....n. Where discrete interval $[0,1]_h$ is the set of points $\{0, h......Nh\}$ for a function *f* and two distinct points x_1, x_2 in it domain the first difference is defined by

$$[x_1, x_2]_f = \frac{[f(x_1) - f(x_2)]}{x_1 - x_2} \tag{3.2}$$

For convenience, we write $f^{(1)}$ for $D_h^{\{1\}} f$ and $w(f, p)$ for modules of continuity of *f*, the discrete norms of a function *f* over the interval $[0, 1]_h$ is defined by

$$\| f \| = \max_{x \in [0,1]_h} | f(x) | \tag{3.3}$$

We shall obtain in the following the bound of error function $e(x) = s(x,h) - f(x)$ over the discrete interval $[0,1]_h$.

Theorem 3.1

Suppose $s(x,h)$ is the discrete quartic spline interpolant of Theorem 2.1. Then

$$\| e(x) \| \le k(P,h)\, w(f, p) \tag{3.4}$$

$$\| e(x_i) \| \le y(h) K^*(P,h)\, w(f, p) \tag{3.5}$$

$$\| e^{(1)}(x) \| \le K_1(P,h)\, w(f, p) \tag{3.6}$$

Where K (p, h), K*(P,h) and $K_1(P,h)$ are some positive functions of p and h.

Proof

Writing $f(x_i) = f_i$ equation 3.1 may be written as

$$A(h).(e(x_i)) = F_i(h) - A(h)(f_i) = L_i(f) \quad \text{(Say)} \tag{3.7}$$

When we replace $s_i(h)$ by $e_i(x_i) = s(x_i, h) - f_i$ (3.8)

We need the following Lemma due to Lyche [4, 5], to estimate inequality (3.5).

Lemma 3.1

Let $\{a_i\}_{i=1}^{m}$ and $\{b_j\}_{j=1}^{n}$ be given sequence of non negative real numbers such that

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Then for any real valued function *f*, defined on discrete interval $[0, 1]_h$, we have

$$\left|\sum_{i=1}^{m} a_i \left[n_{e_0}, x_{e_1}, \ldots\ldots x_{i_k}\right]_f - \sum_{j=1}^{n} b_j \left[y_{i_0}, y_{j_1} \ldots . y_{j_k}\right] f\right|$$

$$\le w\left(f^{(k)}, |1-Kh|\right)\frac{\sum a_i}{K!} \quad (3.9)$$

Where $x_{j_k}, y_{j_k} \in [0,1]_h$ for relevant value of i, j and k. We can write the equation (3.9) is of the form of error function as follows

$$P_1^3\left[\left\{H\left(\frac{92}{8},2\right)P_{i-1}^2 + H\left(\frac{2326}{189},\frac{-32}{3}\right)h^2\right\}e_{i-1}\right.$$

$$+\left[P_1^3\left\{H\left(\frac{-4}{27},\frac{13}{6}\right)P_{i-1}^2 + H\left(\frac{17}{54},\frac{16}{3}\right)h^2\right\}\right.$$

$$\left.+P_{i-1}^3\left\{H\left(\frac{2}{9},\frac{4}{3}\right)P_i^2 - H\left(\frac{26}{27},\frac{16}{3}\right)h^2\right\}\right]e_i$$

$$+P_{i-1}^3\left[H\left(\frac{1}{162},\frac{1}{6}\right)P_i^2 + H\left(\frac{-7}{54},\frac{8}{3}\right)h^2\right]e_{i+1} = R_i(f)$$

Where $R_i(f) = F_i - P_i^3\left[H\left(\frac{92}{81},2\right)P_{i-1}^2 + H\left(\frac{2326}{189}-\frac{32}{3}\right)h^2\right]f_{i-1}$

$$-P_i^3\left[H\left(\frac{-4}{27},\frac{13}{6}\right)P_{i-1}^2 + H\left(\frac{-17}{54},\frac{16}{3}\right)h^2\right]$$

$$-P_{i-1}^3\left[H\left(\frac{-2}{9},\frac{4}{3}\right)P_i^2 - H\left(\frac{26}{27},\frac{16}{3}\right)h^2\right]f_i$$

$$-P_{i-1}^3\left[H\left(\frac{-1}{162},\frac{1}{6}\right)P_i^2 + H\left(\frac{7}{54},\frac{8}{3}\right)h^2\right]f_{i+1} \quad (3.10)$$

Writing equation (3.9) is the form of divided different and using Lemma 3.1 given by Lyche [5]

$$|R_i(f)| = \left[-P_i^3 P_{i-1}\left[\alpha_{i-1},\beta_{i-1}\right]_f\left\{H\left(\frac{2}{9},\frac{-7}{4}\right)P_{i-1}^2 + h^2 H\left(\frac{43}{36},-8\right)\right\}\right.$$

$$-H\left(\frac{11}{81},\frac{-1}{3}\right)\left[\alpha_i, x_i\right]P_i^3 P_{i-1}^3 + P_i^3 P_{i-1}\left[\beta_{i-1_i}, x_i\right]_f\left\{H\left(\frac{10}{81},\frac{-1}{12}\right)\right.$$

$$P_{i-1}^2 + h^2\, H\left(\frac{61}{108},\frac{8}{3}\right)\Big\} + [x_{i-1}, x_i]_f\ \left\{H\left(\frac{-8}{81},-2\right)P_{i-1}^2 + h^2 H\left(\frac{-22}{27},\frac{-32}{3}\right)\right\} P_i^3 P_{i-1}$$

$$-\frac{1}{9}P_i^3 P_{i-1}^3\,[\gamma_i - h, \gamma_i + h] + p_i^3\, P_{i-1}\, H\left(\frac{2}{9},\frac{13}{9}\right)[\gamma_{i-1} - h, \gamma_{i-1} + h]_f$$

$$+ P_{i-1}^3\, P_i\left[\left\{H\left(\frac{-16}{243},0\right)P_i^2 + H\left(\frac{-80}{81},\frac{64}{18}\right)h^2\right\}\right.$$

$$[\alpha_i, \beta_i]_f - \left\{H\left(\frac{-5}{81},\frac{-1}{9}\right)P_i^2 + H\left(\frac{26}{81},\frac{16}{9}\right)h^2\right\}$$

$$[x_i, \alpha_i]_f\,] + \left\{H\left(\frac{1}{243},\frac{-1}{9}\right)p_1^2 + H\left(\frac{7}{81},\frac{-16}{9}\right)h^2\right\}$$

$$[\alpha_i, x_{i+1}]_f - \left(\frac{-11}{9}h^2\right)\ [\gamma_i - h, \gamma_i + h]_f\,]$$

$$\Rightarrow |L_i(f)| = \left|\sum_{i=1}^{5} a_i\,[x_{i_0}, x_{i_1}]_f - \sum_{j=1}^{5} b_j\,[y_{j_0}, y_{j_1}]_f\right| \tag{3.11}$$

$$\le w\left(f^{(1)}, P\right) = \left(\sum_{i=1}^{5} a_i = \sum_{j=1}^{5} b_j\right)$$

$$= P_i^3\, P_{i-1}\left[H\left(\frac{20}{81},\frac{-25}{12}\right)P_{i-1}^2 + h^2 H\left(\frac{43}{36},-8\right)\right]$$

$$+ P_i P_{i-1}^3\left[H\left(\frac{-5}{81},\frac{-1}{9}\right)P_i^2 + H\left(\frac{-73}{81},\frac{-16}{9}\right)h^2\right] \tag{3.12}$$

Where

$$a_1 = P_1^3 P_{i-j}\left[H\left(\frac{10}{81},\frac{-1}{12}\right)P_{i-1}^2 + h^2 H\left(\frac{61}{108},\frac{8}{13}\right)\right]$$

$$a_2 = P_1^3 P_{i-1}\left[H\left(\frac{-8}{81},-2\right)P_{i-1}^2 + H\left(\frac{-22}{27},\frac{-32}{3}\right)h^2\right]$$

$$a_3 = P_1^3 P_{i-1}\left[H\left(\frac{2}{9},\frac{13}{9}\right)\right]$$

$$a_4 = P_{i-1}^3 P_i \left[H\left(\frac{-16}{243}, 0\right) P_1^2 + H\left(\frac{-80}{81}, \frac{64}{18}\right) h^2 \right]$$

$$a_5 = P_{i-1}^2 P_i \left[H\left(\frac{1}{243}, \frac{-1}{9}\right) P_1^2 + H\left(\frac{7}{81}, \frac{-16}{9}\right) h^2 \right]$$

$$b_1 = P_1^3 P_{i-1} \left[H\left(\frac{2}{9}, \frac{-7}{4}\right) P_{i-j}^2 + H\left(\frac{43}{36}, -8\right) h^2 \right]$$

$$b_2 = P_1 P_{i-1}^3 H\left(\frac{11}{81}, \frac{-1}{3}\right)$$

$$b_3 = -\frac{1}{9} p_i^3 p_{i-1}^3$$

$$b_4 = p_{i-1}^3 p_i \left[H\left(\frac{-5}{81}, \frac{-1}{9}\right) p_i^2 + H\left(\frac{26}{81}, \frac{16}{9} h^2\right) h^2 \right]$$

$$b_5 = -\frac{11}{9} h^2 \, p_{i-1}^3 p_i$$

and $x_{1_0} = \beta_{i-1} = y_{1_1}, x_{1_1} = x_1 = x_{2_1} = y_{4_1}$

$x_{2_0} = x_{i-j}, x_{3_0} = y_{i-1} - h,$

$x_{3_1} = \gamma_{i-1} + h$

$y_{1_0} = \alpha_{i-1}, y_{2_0} = \alpha_i = y_{4_1} = x_{4_0} = x_{5_0}$

$y_{3_0} = \gamma_i - h = y_{5_0}, y_{3_1} = \gamma_i + h =_{5_1}$

$x_{4_1} = \beta_i, x_{5_1} = x_{i+1}$

Now using the equation (3.10) and (3.9) in (3.8).

$$\| e(x_i) \| \le y(h) K^*(P,h) h\left(f^{(1)}, p\right) \quad (3.13)$$

This is the inequality (3.5) of Theorem 3.1.

To obtain inequality (3.4) of Theorem 3.1. Writing equation (2.5) in the form of error function as follows:

$$e(x) = e_{i-1} Q_4(t) + e_i Q_5(t) + M_i(f)$$

Where $M_i(f) = f(\alpha_i) Q_1(t) + f(\beta_i) Q_2(t)$

$$+\,p_{i-1}\,f^{(1)}(\gamma_i)Q_3(t)+f_{i-1}\,Q_4(t)+f_iQ_5(t)-f(x) \tag{3.14}$$

Again, we write $M_i(f)$ in form of Divided difference and using Lemma 3.1, we get

$$|M_i(f)|\le w\left(f^{(1)},p\right)\sum_{i=1}^{3}a_i=\sum_{j=1}^{2}b_j$$

$$a_1=P_i\left[\left(\frac{1}{36}+\frac{1}{4}h^2\right)t+\left(\frac{-1}{3}+\frac{3}{4}h^2\right)t^2\quad+\left(\frac{29}{36}-4h^2\right)t^3+\left(\frac{-1}{2}+3h^2\right)t^4\right]$$

$$a_2=P_i\left[\left(\frac{1}{9}-\frac{2}{3}h^2\right)t+\left(\frac{-19}{18}-\frac{1}{6}h^2\right)t^2+\left(\frac{13}{27}+\frac{8}{3}h^2\right)t^3+\left(\frac{5}{9}-2h^2\right)t^4\right]$$

$$q_3=P_i\left[-\frac{1}{9}t+\frac{2}{3}t^2+\frac{11}{9}t^3+\frac{2}{3}t^4\right]$$

$$b_1=P_i\left[\left(\frac{1}{324}+\frac{1}{12}h^2\right)+t^2\left(\frac{-5}{108}+\frac{59}{36}h^2\right)-\left(\frac{-7}{108}+\frac{4}{3}h^2\right)t^3+\left(\frac{1}{18}-h^2\right)t^4\right]$$

$$b_2=P_i\left[t\left(\frac{8}{27}-\frac{1}{3}h^2\right)\right]$$

and $x_{1_0}=\alpha_i,\ x_{1_1}=\beta_i$

$x_{2_0}=x_{i-1},x_{2_1}=\beta_i$

$x_{3_0}=\gamma_i-h,\ \gamma_{3_1}=\gamma_1+h$

$y_{1_0}=\beta_i,\ y_{1_1}=n_i$

$y_{2_0}=x_{i-j},\ y_{2_1}-x$

Thus

$$\sum_{i=1}^{3}a_i=\sum_{j=1}^{2}b_j=P_i\left[\left(\frac{1}{36}-\frac{5}{12}h^2\right)t+\left(\frac{2}{81}-\frac{7}{12}h^2\right)t^2+\left(\frac{7}{108}-\frac{4}{3}h^2\right)t^3\right.$$
$$\left.+\left(\frac{-5}{18}+h^2\right)t^4\right]$$

using (3.5) and(3.13) in (3.14), we get inequality (3.4) of theorem 3.1.

We now proceed to obtain bound of $e^{\{1\}}(x)$

$$P_is_i^{\{1\}}(x)=f(\alpha_i)Q_1^{\{1\}}(t)+f(\beta_i)Q_2^{\{1\}}(t)$$

$$+P_i D_h^{\{1\}} f(\gamma_i) Q_3^{\{1\}}(t) + s_i(x) Q_4^{\{1\}}(t) + s_{i+1} Q_5^{\{1\}}(t) \tag{3.15}$$

$$A_i P_i e^{\{1\}}(x) = e_{i-1} Q_4^{\{1\}}(t) + e_i Q_5^{\{1\}}(t) + U_i(f) \tag{3.16}$$

Where $U_i(f) = f(\alpha_i) Q_1^{\{1\}}(t) + f(\beta_i) Q_2^{\{1\}}(t) + P_i D_h^{\{1\}} f(y_i) Q_3^{\{1\}}(t)$

$$+ f_{i-1} Q_4^{\{1\}}(t) + f_i Q_5^{\{1\}}(t) - AP_i f^{\{1\}}(x)$$

By using Lemma 3.1, we get

$$|U_i(f)| \le w\left(f^{(1)}, P\right) \sum_{i=1}^{3} a_i = \sum_{j=1}^{2} b_j = P_i \left[H\left(\frac{-1}{324}, \frac{-7}{12}\right) \right.$$

$$\left. + H\left(\frac{2}{3}, \frac{-3}{2} h^2\right) t + H\left(\frac{-29}{36}, 4\right)\left(3t^2 + h^2\right) + H\left(\frac{1}{2}, -3\right) 4t\left(t^2 + h^2\right) \right] \tag{3.17}$$

Where $a_1 = P_i \left[H\left(\frac{1}{9}, \frac{-2}{3}\right) - H\left(\frac{58}{81}, \frac{1}{3}\right) t + \left(3t^2 + h^2\right) \right.$

$$\left. H\left(\frac{157}{27}, \frac{8}{3}\right) - 4H\left(\frac{2}{4}, 2\right) t\left(t^2 + h^2\right) \right]$$

$$a_2 = p_i \left[H\left(\frac{-1}{324}, \frac{1}{12}\right) + tH\left(\frac{4}{81}, \frac{7}{6}\right) + \left(3t^2 + h^2\right) \right.$$

$$\left. H\left(\frac{-23}{108}, \frac{4}{3}\right) + 4t\left(t^2 + h^2\right) H\left(\frac{1}{18}, -1\right) \right]$$

$$a_3 = p_i \left[\frac{-1}{9} + \frac{4}{3} t - \frac{1}{4}\left(3t^2 + h^2\right) + \frac{8}{3} t\left(t^2 + h^2\right) \right]$$

$$b_1 = p_i \left[H\left(\frac{-1}{36}, \frac{-1}{4}\right) + H\left(\frac{2}{3}, \frac{-3}{2}\right) t + H\left(\frac{-29}{36}, \frac{12}{3}\right) \right.$$

$$\left. \left(3t^2 + h^2\right) + 4H\left(\frac{1}{2}, -3\right) t\left(t^2 + h^2\right) \right]$$

$b_2 = p_i H\left(\frac{2}{81}, \frac{-1}{3}\right)$ and

$$x_{1_0} = x_i, x_{1_1} = \beta_i = x_{2_1}$$

$$x_{2_2} = x_{i+1}, x_{3_0} = \gamma_i - h, x_{3_1} = \gamma_i + h$$

$$y_{1_0} = \alpha_i, y_{1_1} = \beta_i$$

$$y_{2_0} = x - h, y_{2_1} = x + h$$

By using (3.5), (3.13) and (3.17) in (3.16), we get inequality (3.6) of Theorem 3.1.

REFERENCES

1. Dikshit, H.P. and Powar, P.L. Area Matching interpolation by discrete cubic splines, Approx Theory and Application Proc.Ind. Cont.St. Jhon's New Foundland (1984), 35-45.
2. Dikshit, H.P. and Rana, S.S. Discrete Cubic Spline Interpolation with a non-uniform mesh, Rockey Mount Journal Maths 17 (1987), 700-718.
3. Boneva, K. Kandall D, and Stefanov, I. Spline transformation. J.Roy. Statis, Soc, Ser B 33 (1971), 1-70.
4. Lyche, T. Discrete Cubic Spline Interpolation, Report RRS, University of Cyto 1975.
5. Lyche, T. Discrete cubic spline interpolation, BIT 16, (1978), 281-290.
6. Mangasarian, O.L. and Schumaker, L.L. Discrete spline via Mathematical Programming SIAMJ Controla (1971), 174-183.
7. Mangasarian, O.L. and Schumaker L.L. Best Summation formula and Discrete Splines Num. 10 (1973).
8. Rana, S.S. Local Behavior of the first difference of discrete cubic spline Interpolation, Approx. Theory and Appl. 8(4), 120-127.
9. Rana, S.S. and Dubey, Y.P. Local behavior of the deficient discrete cubic spline Interpolation, J. Approx. Theory 86 (1996), 120-127.
10. Rana, S.S. and Dubey, Y.P. Best error bounds for quantic spline interpolation, Indian Journal of Pure and Appl. Mathematics 28 (2000), 1337-1344.